



Recovering the initial state of an infinite-dimensional system using observers[☆]

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ABSTRACT

Let A be the generator of a strongly continuous semigroup \mathbb{T} on the Hilbert space X , and let C be a linear operator from $\mathcal{D}(A)$ to another Hilbert space Y (possibly unbounded with respect to X , not necessarily admissible). We consider the problem of estimating the initial state $z_0 \in \mathcal{D}(A)$ (with respect to the norm of X) from the output function $y(t) = C\mathbb{T}_t z_0$, given for all t in a bounded interval $[0, \tau]$. We introduce the concepts of estimatability and backward estimatability for (A, C) (in a more general way than currently available in the literature), we introduce forward and backward observers, and we provide an iterative algorithm for estimating z_0 from y . This algorithm generalizes various algorithms proposed recently for specific classes of systems and it is an attractive alternative to methods based on inverting the Gramian. Our results lead also to a very general formulation of Russell's principle, i.e., estimatability and backward estimatability imply exact observability. This general formulation of the principle does not require \mathbb{T} to be invertible. We illustrate our estimation algorithms on systems described by wave and Schrödinger equations, and we provide results from numerical simulations.

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1. Introduction

In many areas of science and engineering it is important to estimate the initial (or the final) state of a linear distributed parameter system (DPS) from its input and output functions measured over some finite time interval. In oceanography and meteorology this problem is called *data assimilation*, see for example Auroux and Blum (2005, 2008), Le Dimet, Shutyaev, and Gejadze (2006), Teng, Zhang, and Huang (2007) or Zou, Navon, and Le Dimet (1992). Such a problem also arises in the context of medical imaging by *impedance-acoustic tomography*; see for instance Gebauer and Scherzer (2008) and the review paper by Kuchment and Kunyansky (2008). The estimation of the initial state can also be regarded as the main step in solving inverse source problems, see Alves, Silvestre, Takahashi, and Tucsnak (2009). An infinite-dimensional system is called *exactly observable in time τ* if the problem of estimating the initial state from data measured over a time interval of length τ is well-posed (see Section 2 for precise definitions).

Suppose now that we have an exactly observable linear DPS. The actual formula for expressing the initial state from the measured segment of the input and output functions involves inverting the Gramian operator of the system (see, for instance, Tucsnak and Weiss (2009, Section 6.1)), and this may be numerically very challenging.

On the other hand, if we find a *stabilizing output injection operator* for the system (its existence follows from exact observability), then we can use a numerical solver to implement an *observer* for the DPS in order to estimate its *current* state from the measured inputs and outputs of the system (see Section 2 for details). The estimate improves as time goes by, more precisely, the estimation error tends to zero as time goes to infinity. From the final state we can, in principle, recover the initial state if the dynamics are time reversible. However, this may increase the estimation error and moreover, sometimes we do not have a very long (in time) segment of output data to operate on.

The purpose of this paper is to describe a way in which we can estimate the initial state of a linear DPS by operating only on a finite segment of output data. In short, the idea is to scan the same segment of data back and forth several times, using two observers, one working in forward time, and one in backward time. This idea has appeared in the recent papers (Auroux & Blum, 2005, 2008) where the method has been mathematically justified for finite-dimensional linear systems with full observation and it has been numerically investigated for more general situations (in particular nonlinear systems). A related work is Phung and

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Zhang (2008) where, based on time reversal methods (see e.g. Fink et al. (2000)), the authors develop a method to identify the initial state for Kirchhoff plate equations with distributed observation. The algorithm in this last paper can be shown to be equivalent to a particular case of the algorithm presented in this paper (see Remark 3.11 below). In Auroux and Blum (2005) and Auroux and Blum (2008) the algorithm has been called “back and forth nudging” whereas in Phung and Zhang (2008) it has been called “time reversal focusing”. An abstract formulation of a related algorithm, suitable for skew-adjoint generators and bounded observation operators, has been given in Ito, Ramdani, and Tucsnak (in press).

Linear DPS often have unbounded control and/or observation operators. This is often the consequence of boundary control and/or boundary observation (see Tucsnak and Weiss (2009) for an elementary introduction to this topic). To make our basic ideas more easily understandable, we give in this introductory section a short presentation of the simple particular case when the observation operator is bounded, the output injection operators are also bounded and the semigroup is invertible (i.e., the observed system is time reversible). However, we emphasize that our results do not require the observation and output injection operators to be bounded (not even admissible), and we also do not require the observed system to be time reversible.

Let X and Y be Hilbert spaces, called the *state space* and *output space*, respectively. Let $A : \mathcal{D}(A) \rightarrow X$ be the generator of a strongly continuous group of operators on X . This group describes the (time reversible) dynamics of our system. The system is described by the equations

$$\dot{z}(t) = Az(t), \quad y(t) = Cz(t),$$

with an initial state $z(0) \in X$, where y is the output function. We assume (in this simplified presentation) that $C \in \mathcal{L}(X, Y)$. The operator C is called the *observation operator*. We assume that (A, C) is *estimatable* with bounded injection, which means that there exists an $H \in \mathcal{L}(Y, X)$ such that $A + HC$ generates an exponentially stable semigroup on X . In this case, we can construct an *observer* for our system as follows: the state of the observer is w and it satisfies the differential equation

$$\dot{w}(t) = (A + HC)w(t) - Hy(t),$$

with an initial state $w(0) \in X$. We refer to Curtain and Zwart (1995) for a discussion of observers in this context. If we define the *estimation error* by $e(t) = w(t) - z(t)$, then it is clear (from subtracting the two differential equations) that

$$\dot{e}(t) = (A + HC)e(t),$$

which shows that $e \rightarrow 0$ exponentially, regardless of the initial states of the system and the observer.

Now suppose that (A, C) is *backward estimatable* with bounded injection, which means there exists an $H_b \in \mathcal{L}(Y, X)$ such that $-A + H_b C$ is the generator of an exponentially stable semigroup on X . In this case, we can construct a *backward observer* for our system as follows: the state of the observer is w_b and it satisfies the differential equation

$$\dot{w}_b(t) = (A - H_b C)w_b(t) + H_b y(t),$$

with a final state $w_b(\tau) \in X$. This equation should be solved *backward in time* on $[0, \tau]$, starting from the final state $w_b(\tau)$. If we define the *backward estimation error* by $e_b(t) = w_b(t) - z(t)$, then from subtracting the differential equations we see that

$$\dot{e}_b(t) = (A - H_b C)e_b(t),$$

which shows that

$$e_b(0) = e^{(-A + H_b C)\tau} e_b(\tau).$$

We define one *estimation cycle* as follows: with the data $y(t)$ given for $t \in [0, \tau]$, we choose an initial state $w(0)$ for the observer and run it over the interval $[0, \tau]$, obtaining an estimate $w(\tau)$ for the unknown state $z(\tau)$. Putting $w_b(\tau) = w(\tau)$, we run the backward observer to obtain an estimate $w_b(0)$ for the unknown state $z(0)$. The estimation error at the end of such a cycle is

$$e_b(0) = e^{(-A + H_b C)\tau} e^{(A + HC)\tau} e(0).$$

If we run N estimation cycles ($N \in \mathbb{N}$) then the estimation error at the end will be

$$e_b^{(N)}(0) = [e^{(-A + H_b C)\tau} e^{(A + HC)\tau}]^N e^{(0)}(0).$$

Since the semigroups generated by $A + HC$ and $-A + H_b C$ are exponentially stable, for $\tau > 0$ large enough, we have

$$\|e^{(-A + H_b C)\tau} e^{(A + HC)\tau}\| < 1.$$

For such τ , repeating estimation cycles leads to a rapid convergence of the estimation error to zero.

Notice that it follows from the above argument that under the stated assumptions (all the operators apart from A are bounded, estimatability and backward estimatability), the original system is exactly observable. This is a dual version of *Russell's principle*, which was originally stated in Russell (1974, 1978) and has been used in many references, such as Komornik (1992). A rigorous and general statement and proof of the principle can be found in the little known conference paper Rebarber and Weiss (1997). The precise statement of a more general version of the dual Russell's principle (not confined to bounded operators and not requiring that the semigroup is invertible) will be given in Proposition 3.3. An even more general version will be given in Remark 3.5.

The paper is organized as follows: in Section 2, we recall some basic facts about admissibility, observability and observers. The iterative reconstruction algorithm is described in Section 3 in a general setting and then applied in Section 4 to a particular class of non reversible second order systems with unbounded observation. Finally, Section 5 provides 2D numerical examples illustrating the efficiency of the proposed reconstruction method.

2. Background on admissibility, observability and observers

First we give some technical background about linear DPS. More precisely, we recall some simple facts about admissible observation and control operators, exact observability and estimatability. There is a large literature on admissibility and we refer to Chapter 5 of Tucsnak and Weiss (2009) for an elementary introduction and for references and historical comments. Here we keep the discussion to the minimum that is needed. The concept of estimatability is one of the infinite-dimensional generalizations of the well-known concept of detectability used in finite-dimensional control theory. It is much less well-known than admissibility, and our basic reference for this concept is Weiss and Rebarber (2000).

Let X and Y be Hilbert spaces and assume that $A : \mathcal{D}(A) \rightarrow X$ is the generator of a strongly continuous semigroup \mathbb{T} on X . We define the Hilbert space $X_1 = \mathcal{D}(A)$, with the norm

$$\|z_0\|_1 = \|(\beta I - A)z_0\| \quad \forall z_0 \in \mathcal{D}(A),$$

where β is an arbitrary fixed element of the resolvent set $\rho(A)$. Regardless of the choice of β , the above norm is equivalent to the graph norm.

Let $C \in \mathcal{L}(X_1, Y)$. For every $\tau > 0$ we define the operator $\Psi_\tau : X_1 \rightarrow L^2([0, \tau]; Y)$ by

$$(\Psi_\tau z_0)(t) = C\mathbb{T}_t z_0 \quad \forall t \in [0, \tau], z_0 \in \mathcal{D}(A).$$

We call C an *admissible observation operator* for \mathbb{T} if for some (hence, for every) $\tau > 0$, Ψ_τ has a continuous extension to X , i.e., $\Psi_\tau \in \mathcal{L}(X, L^2([0, \tau]; Y))$. If this is the case, then there exists a

unique continuous operator $\Psi : X \rightarrow L^2_{loc}([0, \infty); Y)$ such that for every $\tau > 0$ and $z_0 \in X$, $\Psi_\tau z_0$ is the restriction of Ψz_0 to the interval $[0, \tau]$. The Laplace transform of $y = \Psi z_0$ is $\hat{y} = C(sI - A)^{-1}z_0$, for all $z_0 \in X$ and for all $s \in \mathbb{C}$ for which $\text{Re } s$ is larger than the growth bound of \mathbb{T} .

With X, A as above and $C \in \mathcal{L}(X_1, Y)$, the pair (A, C) is called *exactly observable in time* $\tau > 0$ if there exists $k_\tau > 0$ such that

$$\int_0^\tau \|C\mathbb{T}_t z_0\|^2 dt \geq k_\tau^2 \|z_0\|^2 \quad \forall z_0 \in \mathcal{D}(A).$$

The pair (A, C) is called *exactly observable* if it is exactly observable in some time $\tau > 0$.

Note that in the above definition we have not assumed that C is admissible. This is different from the terminology adopted in Tucsnak and Weiss (2009), where admissibility is a part of the definition of exact observability (in time τ).

Let X_{-1} be the completion of X with respect to the norm

$$\|z_0\|_{-1} = \|(\beta I - A)^{-1}z_0\| \quad \forall z_0 \in X,$$

where β is as before. This space may be regarded as the dual of $\mathcal{D}(A^*)$ with graph norm, with respect to the pivot space X . We have $X_1 \subset X \subset X_{-1}$, densely and with continuous embeddings. The semigroup \mathbb{T} can be extended to a strongly continuous semigroup acting on X_{-1} , whose generator is an extension of A , having the domain X . Notationally, we shall not distinguish between \mathbb{T} and its extension, or between A and its extension. Let U be another Hilbert space and let $B \in \mathcal{L}(U, X_{-1})$. For every $\tau > 0$ we define the operator $\Phi_\tau : L^2([0, \tau]; U) \rightarrow X_{-1}$ by

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma.$$

We call B an *admissible control operator* for \mathbb{T} if for some (hence, for every) $\tau > 0$, the range of Φ_τ is contained in X . If this is the case, then Φ_τ is bounded from $L^2([0, \tau]; U)$ to X . If $u \in L^2_{loc}([0, \infty); U)$ then by $\Phi_\tau u$ we mean Φ_τ applied to the truncation of u to $[0, \tau]$. If B is admissible then $\Phi_\tau u$ is a continuous X -valued function. Moreover, for any $u \in L^2_{loc}([0, \infty); U)$ and any $z_0 \in X$, the function $z(t) = \mathbb{T}_t z_0 + \Phi_t u$ satisfies $\dot{z}(t) = Az(t) + Bu(t)$ in the sense that it satisfies (for all $\tau \geq 0$) the integral equation

$$z(\tau) - z(0) = \int_0^\tau [Az(t) + Bu(t)] dt. \tag{2.1}$$

In fact, the function z defined earlier is the unique solution of the above integral equation that satisfies the initial condition $z(0) = z_0$.

Definition 2.1. Let A be the generator of a strongly continuous semigroup \mathbb{T} on X and let $C \in \mathcal{L}(X_1, Y)$. The pair (A, C) is *estimatable* if the following conditions hold:

- (1) There exists an operator $A^K : \mathcal{D}(A^K) \rightarrow X$ that generates an exponentially stable semigroup \mathbb{T}^K on X . We denote by X_{-1}^K the analogue of the space X_{-1} discussed earlier, for the operator A^K .
- (2) There exists $H \in \mathcal{L}(Y, X_{-1}^K)$ that is an admissible control operator for \mathbb{T}^K , such that

$$Ax = A^K x - HCx \quad \forall x \in \mathcal{D}(A). \tag{2.2}$$

Following the finite-dimensional terminology, we call H as above a *stabilizing output injection operator* for (A, C) .

Note that in (2.2), both $A^K x$ and HCx are in X_{-1}^K , but their difference is in X . The Eq. (2.2) can be rewritten equivalently in weak form as follows:

$$\langle Ax, \phi \rangle_X = \langle x, (A^K)^* \phi \rangle_X - \langle Cx, H^* \phi \rangle_Y \quad \forall x \in \mathcal{D}(A), \phi \in \mathcal{D}((A^K)^*). \tag{2.3}$$

Remark 2.2. In the finite-dimensional control literature as well as in the literature on DPS with bounded control and observation

operators, the property defined above is called *detectability*. When dealing with unbounded operators, detectability has several (non-equivalent) generalizations. A more restrictive one has been given in Weiss and Curtain (1997), where there are additional assumptions, for example, (A, H, C) has to be a regular triple. This more restrictive concept is called detectability in Weiss and Curtain (1997). The above definition of estimatability is close to, but not equal to the one in Weiss and Rebarber (2000). Indeed, in Weiss and Rebarber (2000) C is assumed to be admissible for \mathbb{T} , while here we have dropped this requirement. Even for admissible C , the definition of estimatability in Weiss and Rebarber (2000) looks different from the one given here, but they are equivalent, see Propositions 3.3, 3.4, and 4.4 in Weiss and Rebarber (2000).

Remark 2.3. It is known (see Weiss and Rebarber (2000, Section 1)) that if C is admissible for \mathbb{T} , then the exact observability of the pair (A, C) implies its estimatability.

Remark 2.4. Given A and C it is usually not a simple task to find a stabilizing output injection operator H for them. Some approaches involve Riccati equations (see, for instance, Lasiecka (1992)), other approaches involve Gramians (see, for example, Komornik (1997) and Urquiza (2005)). These methods for determining H are computationally expensive in general. If A is skew-adjoint (or close to skew-adjoint) then often we can choose $H = -C^*$. Such an H is called *colocated output injection*, and it is dual to colocated state feedback. Colocated state feedback has been studied a lot in the literature on distributed parameters systems. Among the early papers we mention Haraux (1989) and Slemrod (1974). For the case of skew-adjoint A and bounded C , see our Proposition 3.7. Curtain and Weiss (2006) contains many further references on this topic and it gives rather general sufficient conditions on (A, B) for $-B^*$ to be a stabilizing state feedback operator. Ammari and Tucsnak (2001) considered the case of second order systems without damping.

The above definition of estimatability is clearly a generalization of the one given in the Introduction for bounded operators C and H . It may seem very abstract, but it implies the solvability of a reasonably natural final state estimation problem.

Proposition 2.5. Assume that the pair (A, C) is estimatable and let $\mathbb{T}, A^K, \mathbb{T}^K, H$ be as in Definition 2.1. Take $z_0 \in \mathcal{D}(A)$, let $z \in C([0, \infty); X_1)$ be defined by $z(t) = \mathbb{T}_t z_0$ and let $y \in C([0, \infty); Y)$ be defined by $y = Cz$. Let $w_0 \in X$ and let $w \in C([0, \infty); X)$ be the solution of

$$\dot{w}(t) = A^K w(t) - Hy(t), \quad w(0) = w_0, \tag{2.4}$$

in the sense of (2.1). Then

$$w(t) - z(t) = \mathbb{T}_t^K (w_0 - z_0) \quad \forall t \geq 0. \tag{2.5}$$

In particular, there exist $M \geq 1, \omega > 0$ such that

$$\|w(t) - z(t)\| \leq M e^{-\omega t} \|w_0 - z_0\| \quad \forall t \geq 0.$$

Proof. Clearly for all $t \geq 0$,

$$z(t) - z_0 = \int_0^t Az(\xi) d\xi, \tag{2.6}$$

where z and Az are continuous X -valued functions. The fact that $w \in C([0, \infty); X)$ satisfies (2.4) implies (see (2.1)) that for all $t \geq 0$,

$$w(t) - w_0 = \int_0^t A^K w(\xi) d\xi - \int_0^t Hy(\xi) d\xi. \tag{2.7}$$

In the above formula the two terms in the right-hand side lie in $C([0, \infty), X_{-1}^K)$ but their sum is in $C([0, \infty), X)$. For $t \geq 0$ we

set $e(t) = w(t) - z(t)$. We apply both Eqs. (2.6) and (2.7) to an arbitrary element $\phi \in \mathcal{D}((A^K)^*)$. Then we subtract side by side the resulting equations and use (2.3), to obtain

$$e(t) - e(0) = \int_0^t A^K e(\xi) d\xi \quad \forall t \geq 0.$$

This implies that $e(t) = \mathbb{T}_t^K e(0)$, which is exactly (2.5). The last statement in the proposition follows now from the exponential stability of \mathbb{T}^K . \square

The system (2.4) is called an *observer* for (A, C) (sometimes called a Luenberger observer or a Kalman observer). Such observers for systems with skew-adjoint A , unbounded C and such that (A, C^*, C) determine a regular linear system, were recently discussed in Deguenon, Sallet, and Xu (2006). Other recent papers on the use of observers for the control of linear DPS are Guo and Shao (2009) and Krstic, Guo, Balogh, and Smyshlyaev (2008). Observers for non-linear systems are studied in Bonnabel, Martin, and Rouchon (2008) (in the finite dimensional context), Krstic, Magnis, and Vazquez (2009) and Smyshlyaev and Krstic (2007).

Remark 2.6. Sometimes we have to estimate the initial state of a system that receives an input signal, i.e., it is described by

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \\ y(t) = Cz(t) + Du(t), \\ z(0) = z_0 \in \mathcal{D}(A). \end{cases} \quad (2.8)$$

Here we assume that $B \in \mathcal{L}(U, X)$ and $D \in \mathcal{L}(U, Y)$, where U is another Hilbert space, while $C \in \mathcal{L}(X_1, Y)$, as before. It is not obvious how to make sense of these equations, because C is only defined on $X_1 = \mathcal{D}(A)$. We assume, for the moment, that $u \in \mathcal{H}_{loc}^1((0, \infty); U)$. Then it follows from Tucsnak and Weiss (2009, Theorem 4.1.6) that the first equation in (2.8) has a unique classical solution z and this is a continuous X_1 -valued function of the time t . Thus, the second equation in (2.8) defines a continuous Y -valued function y . We assume that the signal u is available to the observer, defined by

$$\begin{cases} \dot{w}(t) = A^K w(t) + (B + HD)u(t) - Hy(t), \\ w(0) = w_0 \in X. \end{cases}$$

This equation has solutions that are continuous with values in X , because H is an admissible control operator for \mathbb{T}^K . A short computation and proof, along the same lines as the proof of Proposition 2.5, shows that the error satisfies the same Eq. (2.5) as before. This shows that the map $(z_0, w_0, u) \mapsto w(\tau)$ (which is obtained by combining the observer with the system) can be extended by continuity to a bounded operator from $X \times X \times L^2([0, \tau]; U)$ to X (without assuming C to be admissible).

3. Iterations using forward and backward observers

In order to estimate the initial state of a system iteratively using output data from a finite time interval, we need the notion of *backward estimatability*, defined below. In this section (as in the previous one) X and Y denote Hilbert spaces.

Definition 3.1. Let A be the generator of a strongly continuous semigroup \mathbb{T} on X and let $C \in \mathcal{L}(X_1, Y)$. The pair (A, C) is *backward estimatable* if the following conditions hold:

- (1) There exists an operator $A_b^K : \mathcal{D}(A_b^K) \rightarrow X$ that generates an exponentially stable semigroup \mathbb{S}^K on X . We denote by $X_{-1,b}^K$ the analogue of the space X_{-1} discussed earlier, for the operator A_b^K .
- (2) There exists $H_b \in \mathcal{L}(Y, X_{-1,b}^K)$ that is an admissible control operator for \mathbb{S}^K , such that

$$-Ax = A_b^K x - H_b Cx \quad \forall x \in \mathcal{D}(A). \quad (3.1)$$

For (A, C) and H_b as in Definition 3.1, H_b is called a *stabilizing backward output injection operator* for (A, C) .

Note that in the particular situation when \mathbb{T} is invertible (i.e., it can be extended to a group), (A, C) is backward estimatable iff $(-A, C)$ is estimatable.

Proposition 3.2. Assume that A is the generator of a strongly continuous group \mathbb{T} on X , $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for \mathbb{T} and the pair (A, C) is exactly observable. Then (A, C) is forward and backward estimatable.

Proof. By Remark 2.3, (A, C) is estimatable. On the other hand, since A is the generator of a strongly continuous group \mathbb{T} , the exact observability of (A, C) easily implies that $(-A, C)$ is exactly observable. Consequently, (A, C) is also backward estimatable. \square

Proposition 3.3. With the notation and assumptions in Proposition 2.5, suppose that (A, C) is backward estimatable and let A_b^K, H_b be as in (3.1). Let $\tau > 0$ and let w_b be the solution, in a sense similar to (2.1), of the backward problem

$$\begin{cases} \dot{w}_b(t) = -A_b^K w_b(t) + H_b y(t) \quad \text{for } t \leq \tau, \\ w_b(\tau) = w(\tau), \end{cases} \quad (3.2)$$

where w is the solution of (2.4). Then

$$w_b(0) - z_0 = \mathbb{S}_\tau^K \mathbb{T}_\tau^K (w_0 - z_0), \quad (3.3)$$

where \mathbb{S}^K is the semigroup generated by A_b^K . Moreover, the pair (A, C) is exactly observable in any time τ such that $\|\mathbb{S}_\tau^K \mathbb{T}_\tau^K\| < 1$.

Proof. From the definition of z (in Proposition 2.5),

$$z(t) - z(\tau) = - \int_t^\tau Az(\xi) d\xi \quad \forall t \in [0, \tau]. \quad (3.4)$$

The fact that $w_b \in C([0, \infty); X)$ satisfies (3.2) implies that for all $t \in [0, \tau]$,

$$w_b(t) - w_b(\tau) = \int_t^\tau A_b^K w_b(\xi) d\xi - \int_t^\tau H_b y(\xi) d\xi. \quad (3.5)$$

For $t \geq 0$ we set $e_b(t) = w_b(t) - z(t)$. By subtracting side by side (3.4) from (3.5) and using the weak version of (3.1) it follows that

$$e_b(t) - e_b(\tau) = \int_t^\tau A_b^K e_b(\xi) d\xi \quad \forall t \in [0, \tau],$$

which implies that

$$e_b(t) = \mathbb{S}_{\tau-t}^K e_b(\tau) \quad \forall t \in [0, \tau].$$

Since $e_b(\tau) = w(\tau) - z(\tau) = \mathbb{T}_\tau^K (w_0 - z_0)$ (see (2.5)), we obtain (3.3).

Finally we show that the pair (A, C) is exactly observable in every time τ such that $\|\mathbb{S}_\tau^K \mathbb{T}_\tau^K\| < 1$. First note that according to (3.3), if $w_0 = 0$ then

$$w_b(0) = (I - \mathbb{S}_\tau^K \mathbb{T}_\tau^K) z_0.$$

Since $\|\mathbb{S}_\tau^K \mathbb{T}_\tau^K\| < 1$, the operator in the above bracket is invertible, so that

$$z_0 = (I - \mathbb{S}_\tau^K \mathbb{T}_\tau^K)^{-1} w_b(0). \quad (3.6)$$

Note that from (2.4) and (3.2),

$$\begin{cases} w(\tau) = - \int_0^\tau \mathbb{T}_{\tau-s}^K Hy(s) ds, \\ w_b(0) = \mathbb{S}_\tau^K w(\tau) - \int_0^\tau \mathbb{S}_s^K H_b y(s) ds. \end{cases}$$

Substituting the first formula into the second, it follows that

$$w_b(0) = -\mathbb{S}_\tau^K \int_0^\tau \mathbb{T}_{\tau-s}^K Hy(s) ds - \int_0^\tau \mathbb{S}_s^K H_b y(s) ds.$$

Since H and H_b are admissible control operators for \mathbb{T}^K and \mathbb{S}^K respectively, it follows that there exists a constant $K_\tau > 0$ such

that

$$\|w_b(0)\| \leq K_\tau \|y\|_{L^2([0, \tau], Y)}.$$

This together with (3.6) implies the exact observability of (A, C) in time τ . \square

Remark 3.4. The above proposition can be seen as a generalization of the dual form of Russell’s principle (see Russell (1974, 1978)) which asserts that for an operator group, forward and backward stabilizability implies exact controllability. In Chen (1979) an abstract version was given for systems with bounded input operators, and a more general abstract version was given in Rebarber and Weiss (1997) (see the next remark for more detailed comments on this). All these references assume that the system is time-reversible.

Remark 3.5. Following the approach in Rebarber and Weiss (1997), Proposition 3.3 could be generalized in the following way: for every $\alpha \in \mathbb{R}$, we define α -estimatability of (A, C) to mean that $(A - \alpha I, C)$ is estimatable. In this case there exists an operator $A^K : \mathcal{D}(A^K) \rightarrow X$ that generates a semigroup \mathbb{T}^K on X with growth bound $\omega(\mathbb{T}^K) < \alpha$, and there exists $H \in \mathcal{L}(Y, X_{-1}^K)$ such that (2.2) holds. (The proof of this is very easy.)

Backward α -estimatability is defined similarly: (A, C) is backward α -estimatable if the pair $(A + \alpha I, C)$ is backward estimatable. In this case there exists an operator $A_b^K : \mathcal{D}(A_b^K) \rightarrow X$ that generates a semigroup \mathbb{S}^K on X with growth bound $\omega(\mathbb{S}^K) < \alpha$, and there exists $H_b \in \mathcal{L}(Y, X_{-1}^K)$ such that (3.1) holds.

In Proposition 3.3 we could replace the estimatability requirement with α -estimatability and the backward estimatability requirement with backward β -estimatability, as long as $\alpha + \beta \leq 0$ (the conclusions of the proposition remain unchanged).

The proof is a straightforward extension of the proof of Proposition 3.3. The above claim is a generalization of the dual result of Rebarber and Weiss (1997, Theorem 2.1) (it is more general here because we do not require the semigroup \mathbb{T} to be invertible, and we do not require C to be admissible). We mention that in Rebarber and Weiss (1997) there was a sign error in the definition of α -optimizability, the dual concept of α -estimatability: $A + \alpha I$ appeared in place of $A - \alpha I$. There was a similar sign error also in the definition of backward α -optimizability.

In the particular case of time-reversible systems, we have the following.

Proposition 3.6. Let A be the generator of a strongly continuous group \mathbb{T} on X and let $C \in \mathcal{L}(X_1, Y)$. Assume the following.

- (a) (A, C) is estimatable.
- (b) The operators \mathbb{T}_t are uniformly bounded for $t \leq 0$.

Then (A, C) is exactly observable.

Proof. Let \mathbb{T}^K be the exponentially stable semigroup as in Definition 2.1. We argue exactly as in the proof of Proposition 3.3, but with $H_b = 0$, so that $A_b^K = -A$ and $\mathbb{S}_t^K = \mathbb{T}_{-t}$. We have $\|\mathbb{S}_\tau^K \mathbb{T}_\tau^K\| = \|\mathbb{T}_{-\tau} \mathbb{T}_\tau^K\| < 1$ for $\tau > 0$ large enough, so that (3.6) holds, and the proof is finished in the same way as for Proposition 3.3. \square

It follows from this proposition and Proposition 3.2 that for a skew-adjoint A and an admissible C , forward and backward estimatability are equivalent. In the particular case of skew-adjoint generators and bounded C , we have the following.

Proposition 3.7. Assume that A is skew-adjoint on X and let \mathbb{T} be the unitary group on X generated by A . Let $C \in \mathcal{L}(X, Y)$. Then the following assertions are equivalent :

- (i) (A, C) is estimatable.
- (ii) (A, C) is backward estimatable.

(iii) (A, C) is exactly observable.

(iv) $A - C^*C$ generates an exponentially stable group \mathbb{T}^K .

Moreover, if (A, C) is exactly observable in time τ_0 , then the semigroup \mathbb{T}^K defined in (iv) satisfies $\|\mathbb{T}_\tau^K\| < 1$ for every $\tau \geq \tau_0$.

Proof. The equivalence of (i)–(iv) is contained (in dual form) in Theorem 2.3 in Liu (1997). (The equivalence of (i), (ii) and (iii) is also an easy consequence of Propositions 3.2 and 3.6.) The last statement is Lemma 2.2 in Ito et al. (in press). \square

Remark 3.8. Proposition 3.3 implies, in particular, that there exist $M \geq 1, \omega > 0$ (independent of τ) such that

$$\|w_b(0) - z_0\| \leq Me^{-\omega\tau} \|w_0 - z_0\|,$$

so that for large τ , $w_b(0)$ is a good approximation of the initial state z_0 .

The process of computing $w(\tau)$ using an observer and then computing $w_b(0)$ using the backward observer (3.2), as described in Proposition 3.3, may be regarded as one estimation cycle, as described (under much simpler assumptions) in Section 1. In the sequel we discuss iterative repetitions of such estimation cycles.

We give an algorithm allowing the approximation of z_0 provided that τ is such that $\|\mathbb{S}_\tau^K \mathbb{T}_\tau^K\| < 1$. More precisely, for $\tau > 0$ and $z_0 \in \mathcal{D}(A)$, we consider the sequences $(z^{(n)})$ and $(z_b^{(n)})$ in $C([0, \tau], X)$ defined as follows: For every $n \geq 0$, we define $z^{(n)}$ and $z_b^{(n)}$ as the solutions of

$$\begin{cases} \dot{z}^{(n)}(t) = A^K z^{(n)}(t) - Hy(t), \\ z^{(n)}(0) = z_b^{(n-1)}(0) \text{ for } n \geq 1, \\ z^{(0)}(0) = w_0, \end{cases} \quad (3.7)$$

$$\begin{cases} \dot{z}_b^{(n)}(t) = -A_b^K z_b^{(n)}(t) + H_b y(t), \\ z_b^{(n)}(\tau) = z^{(n)}(\tau). \end{cases} \quad (3.8)$$

Here $w_0 \in X$ is an arbitrary initial guess for z_0 .

Proposition 3.9. With the notation and assumptions in Propositions 2.5, 3.3, and using the above notation, for every $n \in \mathbb{N}$ we have

$$z_b^{(n)}(0) - z_0 = (\mathbb{S}_\tau^K \mathbb{T}_\tau^K)^{n+1} (w_0 - z_0). \quad (3.9)$$

In particular if τ is such that $\alpha = \|\mathbb{S}_\tau^K \mathbb{T}_\tau^K\| < 1$, then

$$\|z_b^{(n)}(0) - z_0\| \leq \alpha^{n+1} \|w_0 - z_0\| \quad \forall n \in \mathbb{N}.$$

Proof. The functions $z^{(n)}$ and $z_b^{(n)}$ satisfy the assumptions on w and w_b in Proposition 3.3. It follows from (3.3) that for every integer $n \geq 0$

$$z_b^{(n+1)}(0) - z_0 = \mathbb{S}_\tau^K \mathbb{T}_\tau^K (z^{(n)}(0) - z_0),$$

The above formula clearly implies (3.9). The last conclusion is then straightforward. \square

Remark 3.10. If $w_0 = 0$, then the iterative algorithm for approximating z_0 given in the above proposition is equivalent to performing the inversion in (3.6) using the corresponding Neumann series. We leave it to the reader to verify this fact.

Remark 3.11. In the colocated case $A = -A^*$ and $H = H_b = -C^*$ (discussed in Remark 2.4) the approximation algorithm in Proposition 3.9 can be reformulated in an interesting alternative way such that the forward and backward problems (3.7) and (3.8) become forward problems with the same generator and with zero initial data. For this, first we introduce the time reflection operators \mathfrak{J}_τ for every $\tau > 0$, as follows: if u is a function defined on the

interval $[0, \tau]$, then $(\mathfrak{A}_\tau u)(t) = u(\tau - t)$. For $n \geq 0$ we define the sequences

$$\begin{cases} v^{(2n)} = z^{(n+1)} - z_b^{(n)}, \\ v^{(2n+1)} = \mathfrak{A}_\tau \left(z_b^{(n+1)} - z^{(n+1)} \right). \end{cases} \quad (3.10)$$

We clearly have

$$v^{(n)}(0) = 0 \quad \forall n \geq 0. \quad (3.11)$$

Moreover, it is not difficult to check that for every $n \geq 0$ we have

$$z_b^{(n)} = z_b^{(0)} + \sum_{k=0}^{n-1} \left(v^{(2k)} + \mathfrak{A}_\tau v^{(2k+1)} \right). \quad (3.12)$$

In particular, by using the above relation at time $t = 0$ and the initial condition (3.11), we obtain that

$$z_0 = \lim_{n \rightarrow \infty} z_b^{(n)}(0) = z_b^{(0)}(0) + \sum_{k=0}^{\infty} v^{(2k+1)}(\tau). \quad (3.13)$$

From (3.7)–(3.8) it easily follows that

$$\begin{cases} \dot{v}^{(2n)} = A^K v^{(2n)} + 2C^*(y - Cz_b^{(n)}), \\ \dot{v}^{(2n+1)} = A^K v^{(2n+1)} - 2\mathfrak{A}_\tau C^*(y - Cz_b^{(n+1)}). \end{cases}$$

The alternative algorithm consists of solving the above equations forward in time, using the homogeneous initial conditions (3.11). As $z_b^{(n)}$ can be expressed by (3.12) in terms of $v^{(k)}$ for $k \leq 2n - 1$, we can compute the sequences $v^{(2n)}$ and $v^{(2n+1)}$ recursively. We can obtain an approximation of z_0 by truncating the series in (3.13). The above alternative algorithm has been proposed in Phung and Zhang (2008) in the case of a Kirchhoff plate equation with distributed observation, so that their C is bounded from X to Y .

4. A class of second order systems with unbounded observation

In this section we show that our main results can be applied to a class of second order systems with unbounded observation operators studied, for example, in Weiss and Tucsnak (2003). Most of the systems modeling the linear vibrations of elastic systems can be written in the form described below. Note that the class of systems discussed in this section are not time reversible, in general (see Section 5.2).

First we introduce some notation. Let H be a Hilbert space, and let $A_0 : \mathcal{D}(A_0) \rightarrow H$ be a strictly positive operator. We introduce the scale of Hilbert spaces H_α , $\alpha \in \mathbb{R}$, as follows: for every $\alpha \geq 0$, $H_\alpha = \mathcal{D}(A_0^\alpha)$, with the norm $\|\varphi\|_\alpha = \|A_0^\alpha \varphi\|_H$. The space $H_{-\alpha}$ is defined by duality with respect to the pivot space H as follows: $H_{-\alpha} = H'_\alpha$ for $\alpha > 0$. Equivalently, $H_{-\alpha}$ is the completion of H with respect to the norm $\|\varphi\|_{-\alpha} = \|A_0^{-\alpha} \varphi\|_H$. The operator A_0 can be extended (or restricted) to each H_α , such that it becomes a bounded operator

$$A_0 : H_\alpha \rightarrow H_{\alpha-1} \quad \forall \alpha \in \mathbb{R}. \quad (4.1)$$

Let $C_0 \in \mathcal{L}(H_{\frac{1}{2}}, Y)$, where Y is another Hilbert space. We identify Y with its dual, so that $Y = Y'$. We denote $B_0 = C_0^*$, so that $B_0 \in \mathcal{L}(Y, H_{-\frac{1}{2}})$.

Consider the system described by

$$\frac{d^2}{dt^2} q(t) + A_0 q(t) + \gamma B_0 \frac{d}{dt} C_0 q(t) = 0, \quad (4.2)$$

$$y(t) = \frac{d}{dt} C_0 q(t), \quad (4.3)$$

where $\gamma \geq 0$ is a constant and $t \in [0, \infty)$ is the time. The Eq. (4.2) is understood as an equation in $H_{-\frac{1}{2}}$ (i.e., all the terms are in $H_{-\frac{1}{2}}$). Here, q stands for the displacement field and the term $\gamma B_0 \frac{d}{dt} C_0 q(t)$,

informally written as $\gamma B_0 C_0 \dot{q}(t)$, represents a viscous damping. The state $z(t)$ of this system and its state space X are defined by

$$z(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}, \quad X = H_{\frac{1}{2}} \times H.$$

It is easy to verify that the Eqs. (4.2), (4.3) are equivalent to the following system of first order equations:

$$\begin{cases} \dot{z}(t) = A(\gamma)z(t), \\ y(t) = Cz(t), \end{cases} \quad (4.4)$$

where

$$A(\gamma) = \begin{bmatrix} 0 & I \\ -A_0 & -\gamma B_0 C_0 \end{bmatrix}, \quad (4.5)$$

$$\mathcal{D}(A(\gamma)) = \left\{ \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \mid A_0 \varphi + \gamma B_0 C_0 \psi \in H \right\}, \quad (4.6)$$

$$C : \mathcal{D}(A(\gamma)) \rightarrow Y, \quad C = [0 \ C_0]. \quad (4.7)$$

It is well known that $A(0)$ generates a unitary group in X (see, for instance, Tucsnak and Weiss (2009, Proposition 3.8.7)). For every $\gamma > 0$ the operator $A(\gamma)$ generates a semigroup of contractions (see, for instance, Weiss and Tucsnak (2003, Proposition 5.1)). It is not difficult to check that the adjoint of $A(\gamma)$ is given by

$$\mathcal{D}(A(\gamma)^*) = \left\{ \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \mid A_0 \varphi - \gamma B_0 C_0 \psi \in H \right\} \quad (4.8)$$

$$A(\gamma)^* = \begin{bmatrix} 0 & -I \\ A_0 & -\gamma B_0 C_0 \end{bmatrix}. \quad (4.9)$$

The result below shows that the results in Section 2 can be applied to the system (4.4).

Proposition 4.1. *With the above notation, assume that $A(1)$ is the generator of an exponentially stable semigroup \mathbb{T}^K on X . Then for every $\gamma \geq 0$, the pair $(A(\gamma), C)$ is estimatable and backward estimatable. The corresponding output injection operators are*

$$H = \begin{bmatrix} 0 \\ (\gamma - 1)B_0 \end{bmatrix}, \quad H_b = \begin{bmatrix} 0 \\ -(\gamma + 1)B_0 \end{bmatrix}. \quad (4.10)$$

Proof. Let $X_{-1,1}$ be the dual of $\mathcal{D}(A(1)^*)$ with respect to the pivot space X . In this proof, inner products in H or in Y will not have subscripts, the others will. In order to show that $(A(\gamma), C)$ is estimatable, we prove below that the relation

$$A(1)z = A(\gamma)z + Hcz \quad \forall z \in \mathcal{D}(A(\gamma))$$

holds in $X_{-1,1}$. Let $z = \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A(\gamma))$ and $\phi = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \mathcal{D}((A(1))^*)$. Using (4.5), (4.6), (4.8) and (4.9) we get

$$\begin{aligned} \langle z, (A(1))^* \phi \rangle_X - \langle Cz, H^* \phi \rangle &= \langle A_0^{\frac{1}{2}} \varphi, A_0^{\frac{1}{2}} \eta \rangle + \langle \psi, A_0 \xi - B_0 C_0 \eta \rangle - (\gamma - 1) \langle C_0 \psi, C_0 \eta \rangle \\ &= -\langle A_0 \varphi, \eta \rangle_{H_{-\frac{1}{2}}, H_{\frac{1}{2}}} + \langle A_0^{\frac{1}{2}} \psi, A_0^{\frac{1}{2}} \xi \rangle \\ &\quad - \langle C_0 \psi, C_0 \eta \rangle - (\gamma - 1) \langle C_0 \psi, C_0 \eta \rangle \\ &= \langle A_0^{\frac{1}{2}} \psi, A_0^{\frac{1}{2}} \xi \rangle - \langle A_0 \varphi, \eta \rangle_{H_{-\frac{1}{2}}, H_{\frac{1}{2}}} - \gamma \langle B_0 C_0 \psi, \eta \rangle_{H_{-\frac{1}{2}}, H_{\frac{1}{2}}}. \end{aligned}$$

Since $z \in \mathcal{D}(A(\gamma))$, it follows that

$$\begin{aligned} \langle z, (A(1))^* \phi \rangle_X - \langle Cz, H^* \phi \rangle &= \langle A_0^{\frac{1}{2}} \psi, A_0^{\frac{1}{2}} \xi \rangle - \langle A_0 \varphi + \gamma B_0 C_0 \psi, \eta \rangle \\ &= \langle A(\gamma)z, \phi \rangle_X. \end{aligned}$$

According to Definition 2.1, the above formula implies that the pair $(A(\gamma), C)$ is estimatable with the output injection operator H given by (4.10).

Let $X_{-1,1}^d$ be the dual of $\mathcal{D}(A(1))$ with respect to the pivot space X . In order to show that $(A(\gamma), C)$ is backward estimatable we prove below that the relation

$$A(1)^*z = -A(\gamma)z + H_b Cz \quad \forall z \in \mathcal{D}(A(\gamma)),$$

holds in $X_{-1,1}^d$. Let $z = \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A(\gamma))$ and $\phi = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \mathcal{D}(A(1))$. Using (4.5), (4.6), (4.8) and (4.9), we get

$$\begin{aligned} & \langle z, A(1)\phi \rangle_X - \langle Cz, H_b^* \phi \rangle \\ &= \langle A_0^{\frac{1}{2}} \varphi, A_0^{\frac{1}{2}} \eta \rangle - \langle \psi, A_0 \xi + B_0 C_0 \eta \rangle + (\gamma + 1) \langle C_0 \psi, C_0 \eta \rangle \\ &= \langle A_0 \varphi, \eta \rangle_{H_{-\frac{1}{2}}, H_{\frac{1}{2}}} - \langle A_0^{\frac{1}{2}} \psi, A_0^{\frac{1}{2}} \xi \rangle \\ &\quad - \langle C_0 \psi, C_0 \eta \rangle + (\gamma + 1) \langle C_0 \psi, C_0 \eta \rangle \\ &= \langle A_0^{\frac{1}{2}} (-\psi), A_0^{\frac{1}{2}} \xi \rangle + \langle A_0 \varphi, \eta \rangle_{H_{-\frac{1}{2}}, H_{\frac{1}{2}}} + \gamma \langle B_0 C_0 \psi, \eta \rangle_{H_{-\frac{1}{2}}, H_{\frac{1}{2}}}. \end{aligned}$$

Since $z \in \mathcal{D}(A(\gamma))$, it follows that

$$\begin{aligned} & \langle z, (A(1))^* \phi \rangle_X - \langle Cz, H_b^* \phi \rangle \\ &= \langle A_0^{\frac{1}{2}} (-\psi), A_0^{\frac{1}{2}} \xi \rangle + \langle A_0 \varphi + \gamma B_0 C_0 \psi, \eta \rangle \\ &= -\langle A(\gamma)z, \phi \rangle_X. \end{aligned}$$

According to Definition 2.1, the above formula implies that the pair $(-A(\gamma), C)$ is estimatable, with the output injection operator H_b given by (4.10). \square

By combining the last result and Proposition 3.9 we obtain an algorithm to estimate the initial state of the system (4.2)–(4.3). More precisely, for every $n \geq 0$ we define the sequences $(q^{(n)})$ and $(\dot{q}^{(n)})$ by

$$\begin{cases} \ddot{q}^{(n)}(t) + A_0 \dot{q}^{(n)}(t) + B_0 C_0 \dot{q}^{(n)}(t) = (1 - \gamma) B_0 y, \\ q^{(n)}(0) = q_b^{(n-1)}(0) \quad \text{for } n \geq 1, & q_b^{(0)}(0) = w_0, \\ \dot{q}^{(n)}(0) = \dot{q}_b^{(n-1)}(0) \quad \text{for } n \geq 1, & \dot{q}_b^{(0)}(0) = w_1, \end{cases} \quad (4.11)$$

$$\begin{cases} \ddot{q}_b^{(n)}(t) + A_0 \dot{q}_b^{(n)}(t) - B_0 C_0 \dot{q}_b^{(n)}(t) = -(1 + \gamma) B_0 y, \\ q_b^{(n)}(\tau) = q^{(n)}(\tau), \\ \dot{q}_b^{(n)}(\tau) = \dot{q}^{(n)}(\tau). \end{cases} \quad (4.12)$$

Here $w_0 \in H_{\frac{1}{2}}$ and $w_1 \in H$ are arbitrary initial guesses for q_0 and q_1 .

Recall from Proposition 4.1 that \mathbb{T}^K denotes the semigroup generated by $A(1)$. From Proposition 3.9 we obtain the following.

Corollary 4.2. Assume that \mathbb{T}^K is exponentially stable and let $\tau > 0$ be such that $\|\mathbb{T}_\tau^K\| < 1$. Denote $\alpha = \|\mathbb{T}_\tau^K\|^4$. Then for all $n \in \mathbb{N}$,

$$\begin{aligned} & \left\| q_b^{(n)}(0) - q(0) \right\|_{\frac{1}{2}}^2 + \left\| \dot{q}_b^{(n)}(0) - \dot{q}(0) \right\|_H^2 \\ & \leq \alpha^{n+1} \left(\|w_0 - q(0)\|_{\frac{1}{2}}^2 + \|w_1 - \dot{q}(0)\|_H^2 \right). \end{aligned}$$

The second example in Section 5 shows that the above abstract setting can be applied to a system described by the wave equation in a domain $\Omega \subset \mathbb{R}^n$ with velocity observation on a part of the boundary.

5. Examples and numerical results

In this section we apply our algorithm to reconstruct the initial state for the Schrödinger equation and the wave equation in the unit square $\Omega \subset \mathbb{R}^2$. In the first case, the observation is distributed on a subset of Ω , whereas in the second case the observation is localized on two adjacent sides of Ω .

All the computations were performed on a Power Mac G4 2.8 GHz with 8 GB DDR SDRAM and the algorithms were developed under Matlab.

5.1. The Schrödinger equation

In this subsection we consider the problem of estimating the initial state of a system described by the Schrödinger equation with observation on a part of the domain.

Let Ω denote a rectangular domain in \mathbb{R}^d , $d \geq 1$. We consider the following initial and boundary value problem:

$$\begin{cases} \dot{z} + i\Delta z = 0, & x \in \Omega, t \in [0, +\infty), \\ z = 0, & x \in \partial\Omega, t \in [0, +\infty), \\ z(x, 0) = z_0(x), & x \in \Omega. \end{cases} \quad (5.1)$$

Given a nonempty open subset $\mathcal{O} \subset \Omega$, we consider the output $y = z|_{\mathcal{O}}$. (5.2)

Eqs. (5.1)–(5.2) can be written in the standard form

$$\dot{z} = Az, \quad y = Cz,$$

if we introduce the following spaces and operators:

- the state space is $X = L^2(\Omega)$,
- $A : \mathcal{D}(A) = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega) \rightarrow X$ is the skew-adjoint operator defined by

$$A\varphi = -i\Delta\varphi \quad \forall \varphi \in \mathcal{D}(A), \quad (5.3)$$

- The output space is $Y = L^2(\mathcal{O})$,
- the observation operator $C \in \mathcal{L}(X, Y)$ is defined by

$$C\varphi = \varphi|_{\mathcal{O}} \quad \forall \varphi \in X. \quad (5.4)$$

According to Komornik (1992) the pair (A, C) is exactly observable in any time $\tau > 0$. According to Proposition 3.7 (A, C) is estimatable and backward estimatable. Clearly the same statements hold also for $(-A, C)$. Moreover, the semigroups \mathbb{T}^K and \mathbb{S}^K generated by $A - C^*C$ and $-A - C^*C$, respectively, satisfy $\|\mathbb{T}_\tau^K\| < 1$ and $\|\mathbb{S}_\tau^K\| < 1$ for every $\tau > 0$.

According to Proposition 3.9 the initial state of the system (5.1) can be approximated using the output data (5.2) by the recursive algorithm

$$\begin{cases} \dot{z}^{(n)}(t) = (-i\Delta - \chi_{\mathcal{O}})z^{(n)}(t) + y(t), \\ z^{(n)}(0) = z_b^{(n-1)}(0) \quad \text{for } n \geq 1, \\ z^{(0)}(0) = w_0, \end{cases} \quad (5.5)$$

$$\begin{cases} \dot{z}_b^{(n)}(t) = (-i\Delta + \chi_{\mathcal{O}})z_b^{(n)}(t) - y(t), \\ z_b^{(n)}(\tau) = z^{(n)}(\tau). \end{cases} \quad (5.6)$$

In the above equations we have suppressed the dependence on the position x . $\chi_{\mathcal{O}}$ is the characteristic function of \mathcal{O} and y is extended by zero outside \mathcal{O} . As usual, $w_0 \in X$ is an arbitrary initial guess for z_0 .

In order to show the efficiency of our iterative algorithm, we consider the particular case where Ω is the unit square in \mathbb{R}^2 , $\mathcal{O} = (1/3, 2/3) \times (1/3, 2/3)$, $\tau = 0.2$ (note that τ can be chosen very small, as (A, C) is exactly observable in arbitrarily small time) and the initial data to be recovered is $z_0(x_1, x_2) = 4x(1 - x_1) \sin(5\pi x_2) + i \sin(3\pi x_1) \sin(2\pi x_2)$. We use a Crank–Nicolson scheme in time combined with a finite difference space discretization to simulate the forward and backward systems (5.5) and (5.6). The initial guess w_0 is taken to be zero, the time step is $\Delta t = \tau/300$ and the number of space discretization points is $N_{x_1} = N_{x_2} = 150$ in each direction. Fig. 1 shows the evolution of the relative error (in %) as a function of the number of iterations for a noisy observation with a noise level of 10% (we used a noise obtained by adding random numbers uniformly distributed in $[-\|y\|_{L^\infty}/10, \|y\|_{L^\infty}/10]$ to all entries of $y(t)$). After 10 iterations, the relative error decreases from 19.7% to 0.5%. The error can be further reduced either by reducing the mesh sizes or by using high-order solvers. Each forward or backward iteration of the algorithm required approximately 20 s.

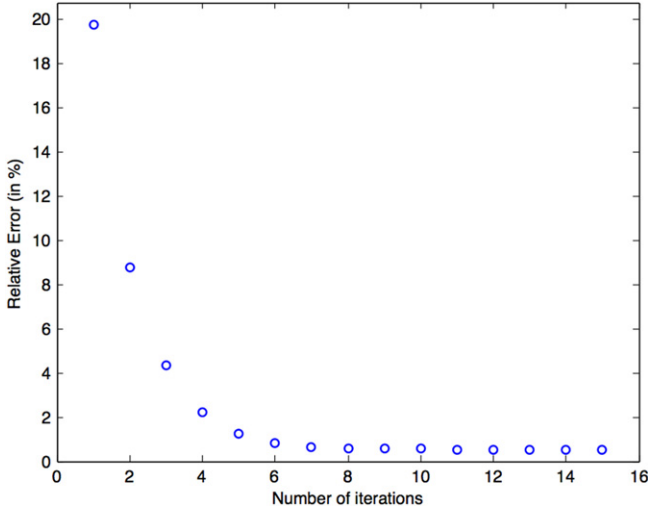


Fig. 1. Relative error (in %) as a function of the number of iterations.

5.2. The wave equation

In this subsection we consider the problem of estimating the initial state of a system described by the wave equation with velocity observation on a part of the boundary.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary Γ and let Γ_0 and Γ_1 be subsets of Γ . Assume that one of the two following assumptions holds:

- (H1): Γ is of class C^∞ , Γ_0 and Γ_1 are relatively open (and also relatively closed) and nonempty subsets of Γ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\Gamma_0 \cup \Gamma_1 = \Gamma$. Moreover, Γ_1 satisfies the geometric optics condition of Bardos, Lebeau, and Rauch (1992).
- (H2): $\Omega = [0, 1]^2$, $\Gamma_0 = (\{1\} \times [0, 1]) \cup ([0, 1] \times \{1\})$ and $\Gamma_1 = \Gamma \setminus \Gamma_0$.

We denote by x the space variable ($x \in \overline{\Omega}$). Consider the system

$$\begin{cases} \ddot{q}(x, t) = \Delta q(x, t) & \text{in } \Omega \times [0, \infty), \\ q(x, t) = 0 & \text{on } \Gamma_0 \times [0, \infty), \\ \frac{\partial q}{\partial \nu}(x, t) + \gamma \dot{q}(x, t) = 0 & \text{on } \Gamma_1 \times [0, \infty), \\ q(x, 0) = q_0(x) & \text{in } \Omega, \\ \dot{q}(x, 0) = q_1(x) & \text{in } \Omega, \\ y(x, t) = \dot{q}(x, t) & \text{on } \Gamma_1 \times [0, \infty), \end{cases} \quad (5.7)$$

where $\gamma \geq 0$ is constant, ν is the unit outward normal to $\partial\Omega$, y is the output function and the functions q_0 and q_1 are the initial state of the system. Note that the above system is not necessarily time-reversible. The easiest counterexample can be found by taking Ω one-dimensional and $\gamma = 1$ (see, for instance, the last paragraph in Weiss and Tucsnak (2003)). Let us also note that the observation operator corresponding to (5.7) is not admissible (see, for instance, Avdonin and Ivanov (1995, p. 220) and Tataru (1998)).

As shown, for instance, in Weiss and Tucsnak (2003), the above system can be written in the form (4.2), (4.3) with the following choice of spaces and operators:

$$H = L^2(\Omega), \quad Y = L^2(\Gamma_1),$$

$$H_1 = \left\{ f \in \mathcal{H}_{\Gamma_0}^1(\Omega) \mid \Delta f \in L^2(\Omega), \frac{\partial f}{\partial \nu} \Big|_{\Gamma_1} = 0 \right\}$$

$$A_0 \varphi = -\Delta \varphi \quad \forall \varphi \in \mathcal{D}(A_0) = H_1$$

where $\mathcal{H}_{\Gamma_0}^1(\Omega) = \{f \in \mathcal{H}^1(\Omega) \mid f|_{\Gamma_0} = 0\}$.

It is well-known that A_0 is self-adjoint, positive and boundedly invertible, see, for instance, Rodriguez-Bernal and Zuazua (1995). Moreover, $H_{\frac{1}{2}} = \mathcal{H}_{\Gamma_0}^{\frac{1}{2}}(\Omega)$.

The operator $C_0 \in \mathcal{L}(H_{\frac{1}{2}}, Y)$ is defined by

$$C_0 f = f|_{\Gamma_1} \quad \forall f \in H_{\frac{1}{2}}.$$

In order to determine $B_0 = C_0^*$ we consider the operator $N \in \mathcal{L}(Y, H_{\frac{1}{2}})$, called the Neumann map, with the following property:

$Nv = g$ if and only if $g \in \mathcal{H}_{\Gamma_0}^1(\Omega)$, $\Delta g = 0$ and $\frac{\partial g}{\partial \nu} = v$ on Γ_1 . As shown in Weiss and Tucsnak (2003, Section 7), we have $B_0 \in \mathcal{L}(U, H_{-\frac{1}{2}})$ and

$$B_0 = C_0^* = A_0 N. \quad (5.8)$$

In order to determine the initial state (q_0, q_1) of (5.7) from the boundary observations y , we consider the sequences $(q^{(n)})_{n \in \mathbb{N}}$ and $(q_b^{(n)})_{n \in \mathbb{N}}$ defined by

$$\begin{cases} \ddot{q}^{(n)}(x, t) = \Delta q^{(n)}(x, t) & \text{in } \Omega \times [0, \infty), \\ q^{(n)}(x, t) = 0 & \text{on } \Gamma_0 \times [0, \infty), \\ \frac{\partial q^{(n)}}{\partial \nu}(x, t) + \dot{q}^{(n)}(x, t) = (1 - \gamma)y(x, t) & \text{on } \Gamma_1 \times [0, \infty), \\ q^{(n)}(x, 0) = q_b^{(n-1)}(x, 0) & \text{for } n \geq 1 \text{ in } \Omega, \\ \dot{q}^{(n)}(x, 0) = \dot{q}_b^{(n-1)}(x, 0) & \text{for } n \geq 1 \text{ in } \Omega, \\ q^{(0)}(x, 0) = w_0(x) & \text{in } \Omega, \\ \dot{q}^{(0)}(x, 0) = w_1(x) & \text{in } \Omega, \end{cases} \quad (5.9)$$

$$\begin{cases} \ddot{q}_b^{(n)}(x, t) = \Delta q_b^{(n)}(x, t) & \text{in } \Omega \times [0, \infty), \\ q_b^{(n)}(x, t) = 0 & \text{on } \Gamma_0 \times [0, \infty), \\ \frac{\partial q_b^{(n)}}{\partial \nu}(x, t) - \dot{q}_b^{(n)}(x, t) = -(1 + \gamma)y(x, t) & \text{on } \Gamma_1 \times [0, \infty), \\ q_b^{(n)}(x, \tau) = q^{(n)}(x, \tau) & \text{in } \Omega, \\ \dot{q}_b^{(n)}(x, \tau) = \dot{q}^{(n)}(x, \tau) & \text{in } \Omega. \end{cases} \quad (5.10)$$

As usual, $w_0 \in H_{\frac{1}{2}}$ and $w_1 \in H$ are arbitrary initial guesses for q_0 and q_1 .

These equations are obtained by specializing to the above choice of A_0 and C_0 the iterative scheme from (4.11) and (4.12). From Corollary 4.2 we obtain.

Corollary 5.1. *If $q_0 \in H_1$, $q_1 \in H_{\frac{1}{2}}$ and one of the assumptions (H1) or (H2) holds, then for τ sufficiently large, there exists $\alpha \in (0, 1)$ such that, for all $n \in \mathbb{N}$,*

$$\begin{aligned} & \left\| \nabla q_b^{(n)}(\cdot, 0) - \nabla q_0 \right\|_{L^2(\Omega)}^2 + \left\| \dot{q}_b^{(n)}(\cdot, 0) - q_1 \right\|_{L^2(\Omega)}^2 \\ & \leq \alpha^{n+1} \left(\left\| \nabla w_0 - \nabla q_0 \right\|_{L^2(\Omega)}^2 + \left\| w_1 - q_1 \right\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

Proof. If (H1) holds, the geometric optics condition insures the exponential stability of (5.7) with $\gamma = 1$, see Bardos et al. (1992). If (H2) holds, the exponential stability of (5.7) with $\gamma = 1$ follows from Cornilleau, Lohéac, and Osses (2010, Remark 2). \square

Concerning the minimal time required in the above corollary, we conjecture that it is given by the maximal time needed by a ray of geometric optics before intersecting Γ_1 . In particular, if (H2) holds, this time should be $2\sqrt{2}$. Nevertheless, there exist many initial states which can be recovered from data from a shorter observation time.

As a numerical illustration of our iterative algorithm, we consider problem (5.7) on the unit square $\Omega = [0, 1]^2$ and with $\gamma = 1$ (non reversible problem). The initial data to be recovered are given by the “bubble” function $q_0(x) = q_1(x) = x(1 - x)y(1 - y)$ and the time of observation is $\tau = 2\sqrt{2}$ which is

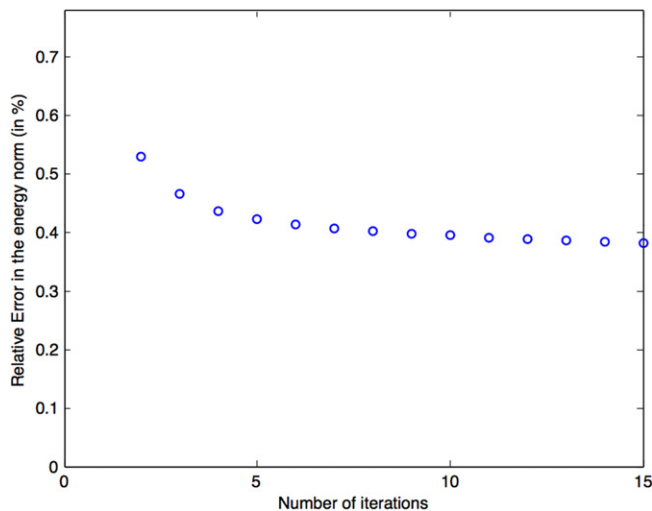


Fig. 2. Relative error (in %) on (q_0, q_1) for the norm in $X = H_{\frac{1}{2}} \times H$ as a function of the number of iterations.

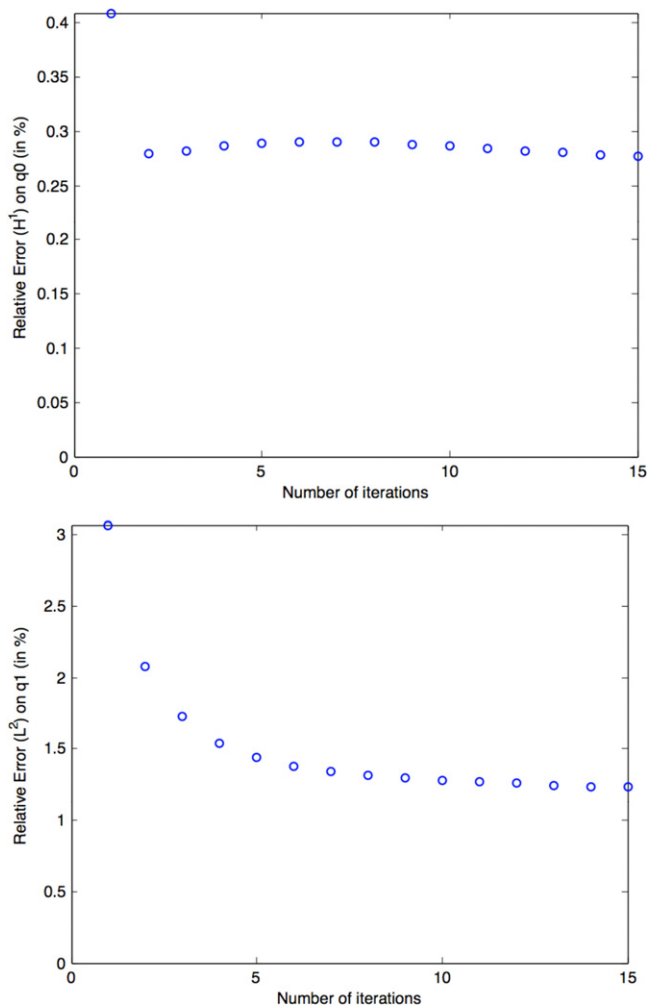


Fig. 3. **Top:** the norm in $H_{\frac{1}{2}}$ of the relative error in q_0 (in %) versus the number of iterations. **Bottom:** the norm in H of the relative error on q_1 (in %) versus the number of iterations.

the exact observability time of the problem. To solve numerically (5.9) and (5.10), we use an explicit centered scheme in time combined with \mathbb{P}^1 -finite elements for the space discretization on a structured mesh. The initial guess (w_0, w_1) in (5.9) is

chosen to be zero (and thus the initial relative error is 100%) and we used a noisy observation (random noise with level 10% obtained by adding random numbers uniformly distributed in $[-\|y\|_{L^\infty}/10, \|y\|_{L^\infty}/10]$ to all entries of $y(t)$). We used 35 points of space discretization in each direction and the time step $\Delta t = 3e - 5$. Each forward or backward iteration of the algorithm requires approximately 30 s.

Fig. 2 shows the decay of the relative error in the energy norm (i.e. for the norm in $X = H_{\frac{1}{2}} \times H$): after 15 iterations, this error is about 0.4%.

Fig. 3 shows the relative errors between the exact and estimated initial data, separately for q_0 and q_1 . After 15 iterations, the relative error obtained is about 0.27% for the approximation of q_0 and about 1.2% for q_1 . As we used a low order solver, a large number of iterations is needed to obtain an accurate approximation of the initial time derivative q_1 , while the second iteration provides a very precise approximation of q_0 . These results can be improved by using high order schemes.

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